Self-Embeddings: from Vaught to Tanaka

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Athens

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- Vaught's proof used *special* models.
- A model $\mathcal{A}$ is special if A can be written as the union of a chain $\langle \mathcal{A}_\alpha : \alpha < |A| \rangle$ such that each $\mathcal{A}_\alpha$ is α^+-saturated.

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- $\mathcal{M}^* \cong \omega^*$ since $\mathcal{M}^*$ and ω^* are elementarily equivalent, special models of the same cardinality.
- With the help of the MacDowell-Specker theorem, and the Löwenheim-Skolem theorem, the above proof shows that every consistent extension of PA has a *countable* model that is isomorphic to an elementary proper initial segment of itself.

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1973, 1977: Alex Wilkie's Contributions

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- **(1)** $\mathrm{Th}_{\Pi_2}(\mathcal{M}) \subseteq \mathrm{Th}_{\Pi_2}(\mathcal{N})$.
- **(2)** *There are arbitrarily high initial segment of* $\mathcal{N}$ *that are isomorphic to* $\mathcal{M}$.

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- **Theorem.** *A countable model $\mathcal{M}$ of Π_2^{PA} is isomorphic to a proper initial segment of itself iff $\mathcal{M}$ is* 1*-tall and* 1*-extendible.*

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- 1-tall means that the set of Σ_1-definable elements of $\mathcal{M}$ is not cofinal in $\mathcal{M}$.
- 1-extendible means that there is an end extension $\mathcal{M}^*$ of $\mathcal{M}$ that satisfies $\mathsf{I}\Delta_0$ and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) = \mathrm{Th}_{\Sigma_1}(\mathcal{M}^*)$.

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- Recall: countable recursively saturated models are resplendent.
- The fact that special models are resplendent was first noted by Chang and Moschovakis (1968).

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- Key idea: in a nonstandard model $\mathcal{M}$ of PA, the Σ_n-types that are realized are precisely the ones coded in the standard system of $\mathcal{M}$.
- The above is the direct consequence of the existence of definable Σ_n-satisfaction predicates for each n.

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- Lipshitz's proof used the Friedman embedding theorem and the MRDP theorem.
- Later (1987) Bonnie Gold refined Lipshitz's aforementioned result by showing:
  Theorem. *If* $\mathcal{M}$ *and* $\mathcal{N}$ *are models of* PA *with* $\mathcal{M} \subseteq_{\text{end}} \mathcal{N}$, *then*:
  $\mathcal{N}$ *is Diophantine correct relative to* $\mathcal{M}$ *iff*
  for every $a \in N \backslash M$ *there is an embedding* $j : \mathcal{N} \to \mathcal{N}$ *such that* $j(N) < a$ *and* $j(m) = m$ *for all* $m \in M$.

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- **Theorem.** *Suppose I is a cut closed under exponentiation that is shared by two nonstandard models $\mathcal{M}$ and $\mathcal{N}$ of* PA *such that*:
  $\mathcal{M}$ *and* $\mathcal{N}$ *have the same I-standard system, and*
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- *Then there is an embedding j of $\mathcal{M}$ onto a proper initial segment of $\mathcal{N}$ such that $j(c) = c$ for all $c \in I$.*

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1981: Jeff Paris' Variation on a Theme by Robert Solovay

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- **Theorem.** *Every countable recursively saturated model of* $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ *is isomorphic to a proper initial segment of itself.*

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- **Theorem.** *Suppose* $\mathcal{M}$ *is a countable model of* PA *and* $a \notin \Delta_1^{\mathcal{M}}$, *then there is a self-embedding of* $\mathcal{M}$ *onto a submodel* $\mathcal{N}$ *such that* $a \notin N$.

- **Theorem.** *Suppose $\mathcal{M}$ is a countable model of* PA *and* $a \notin \Delta_1^{\mathcal{M}}$*, then there is a self-embedding of $\mathcal{M}$ onto a submodel $\mathcal{N}$ such that* $a \notin N$.
- (Marker and Wilkie) In the above, $\mathcal{N}$ can be arranged to be an initial segment of $\mathcal{M}$ if there is no $b > a$ with $b \in \Delta_1^{\mathcal{M}}$.

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- A theory is *rich* if it has a recursive sequence of independent formulae.
- Recursively saturated models of rich theories have canonical standard systems.
- **Theorem.** *The isomorphism type of a countable recursively saturated model* $\mathcal{M}$ *of a rich theory is determined by* $\mathrm{Th}(\mathrm{M})$ *and* $\mathrm{SSy}(\mathrm{M})$.

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$$(\mathcal{M}, a_1, \cdots, a_n, c) \equiv (\mathcal{M}, b_1, \cdots, b_n, d).$$

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- (3) The set of types that are coded in a recursively saturated model of a rich theory are precisely those types that are coded in SSy($\mathcal{M}$).

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- Let d be some fixed nonstandard member of $\mathcal{M}$, and for each $c \in M$, let $Th^{\leq d}(\mathcal{M}_c)$ be the set of sentences of length at most d that are true in $\mathcal{M}_c$, as computed in $\mathcal{M}$.

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- This shows that there is some $e' \in M^* \backslash M$ such that $Th^{\leq d}(\mathcal{M}_e) = Th^{\leq d}(\mathcal{M}^*_{e'})$.
- By the Ehrenfeucht-Jensen theorem we may conclude that there is an isomorphism $f : \mathcal{M}^*_{e'} \to \mathcal{M}_e$. By restricting f to $\mathcal{M}$, we obtain an embedding of into a proper initial segment of itself.

1987: Jean-Pierre Ressayre's Optimal Result

- **Theorem.** *Suppose* $\mathcal{M}$ *is for every countable nonstandard model* $\mathcal{M}$ *of* $I\Sigma_1$ *and for every* $a \in \mathcal{M}$ *there is an embedding* j *of* $\mathcal{M}$ *onto a proper initial segment of itself such that* $j(x) = x$ *for all* $x \leq a$.

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- *Moreover, the above property characterizes countable models of* $I\Sigma_1$ *among countable models of* $\mathrm{I}\Delta_0$.

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- Independently of Ressayre, Dimitracopoulos and Paris showed that every countable nonstandard model of $\mathsf{I}\Sigma_1$ is isomorphic to a proper initial segment of itself.
- They also generalized Lessan's aforementioned result by weakening Π_2^{PA} to $\mathsf{I}\Delta_0 + \exp + \mathrm{B}\Sigma_1$.

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- **(4)** *$f(a) < b$ for all partial $\mathcal{M}$-recursive functions.*

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- Tanaka's motivation for his result was the development of non-standard methods within the confines of the frugal system WKL_0.
- A remarkable application of Tanaka's result appears in the work of Tanaka and Yamazaki, where it is used to show that the construction of the Haar measure (over compact groups) can be implemented within WKL_0 via a detour through nonstandard models.
- This is in contrast to the previously known constructions of the Haar measure whose implementation can only be accommodated within the stronger fragment ACA_0 of second order arithmetic.

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- **Theorem.** *Every countable nonstandard model* $(\mathcal{M}, \mathcal{A})$ *of* WKL_0 *is isomorphic to a proper initial segment* I *of itself in the sense that there is an isomorphism* $\phi : \mathcal{M} \to I$ *such that* ϕ *induces an isomorphism* $\widehat{\phi} : (\mathcal{M}, \mathcal{A}) \to (I, \mathcal{A} \restriction I)$.

Tanaka's Theorem

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- In the above:
  $\mathcal{A} \restriction I := \{A \cap I : A \in \mathcal{A}\}$, and $\widehat{\phi}$ is defined by:
  $\widehat{\phi}(m) = \phi(m)$ for $m \in M$ and
  $\widehat{\phi}(A) = \{\phi(a) : a \in A\}$ for $A \in \mathcal{A}$.
- Moreover, given any prescribed $a \in M$, *there is some* I *and* ϕ *as above such that* $\phi_a(m) = m$ *for all* $m \leq a$.

Thank You

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- $\varepsilon\upsilon\chi\alpha\rho\iota\sigma\tau\acute{\omega}$