Self-Embeddings: Some Recent Results

Ali Enayat
University of Gothenburg

32ème Journées sur les Arithmétiques Faibles

June 25, 2013
Athens

Modern Versions of Friedman's Theorem (1)

Modern Versions of Friedman's Theorem (1)

- Unless stated otherwise, in this talk $\mathcal{M}$ is a countable model of $\mathrm{I}\Sigma_1$, and I is a proper initial segment of $\mathcal{M}$.

Modern Versions of Friedman's Theorem (1)

- Unless stated otherwise, in this talk $\mathcal{M}$ is a countable model of $\mathrm{I}\Sigma_1$, and I is a proper initial segment of $\mathcal{M}$.
- A partial function f from M to M is a *partial* $\mathcal{M}$*-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.

Modern Versions of Friedman's Theorem (1)

- Unless stated otherwise, in this talk $\mathcal{M}$ is a countable model of $\mathrm{I}\Sigma_1$, and I is a proper initial segment of $\mathcal{M}$.
- A partial function f from M to M is a *partial $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem**. *Suppose $c \in M$, and $\{a, b\} \subseteq N$ with $a < b$. The following statements are equivalent*:

Modern Versions of Friedman's Theorem (1)

- Unless stated otherwise, in this talk $\mathcal{M}$ is a countable model of $\mathrm{I}\Sigma_1$, and I is a proper initial segment of $\mathcal{M}$.
- A partial function f from M to M is a *partial $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem**. *Suppose $c \in M$, and $\{a, b\} \subseteq N$ with $a < b$. The following statements are equivalent*:
- **(1)** $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$, *and for every Δ_0-formula $\delta(x, y)$ we have*:

$$\mathcal{M} \models \exists y\ \delta(c, y) \Longrightarrow \mathcal{N} \models \exists y < b\ \delta(a, y).$$

Modern Versions of Friedman's Theorem (1)

- Unless stated otherwise, in this talk $\mathcal{M}$ is a countable model of $\mathrm{I}\Sigma_1$, and I is a proper initial segment of $\mathcal{M}$.
- A partial function f from M to M is a *partial $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem**. *Suppose $c \in M$, and $\{a, b\} \subseteq N$ with $a < b$. The following statements are equivalent*:
- **(1)** $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$, *and for every Δ_0-formula $\delta(x, y)$ we have*:

$$\mathcal{M} \models \exists y\ \delta(c, y) \Longrightarrow \mathcal{N} \models \exists y < b\ \delta(a, y).$$

- **(2)** *There is an initial embedding $j : \mathcal{M} \to \mathcal{N}$ with $j(c) = a$ and $a < j(M) < b$.*

Modern Versions of Friedman's Theorem (2)

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent:*

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent:*
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $j(a) = a$ *and* $a < j(M) < b$.

Modern Versions of Friedman's Theorem (2)

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent:*
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $j(a) = a$ *and* $a < j(M) < b$.
- **(2)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}(\mathcal{M}, a) = \mathrm{Th}(I, a)$.

Modern Versions of Friedman's Theorem (2)

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent:*
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $j(a) = a$ *and* $a < j(M) < b$.
- **(2)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}(\mathcal{M}, a) = \mathrm{Th}(I, a)$.
- **(3)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}_{\Sigma_1}(\mathcal{M}, a) = \mathrm{Th}_{\Sigma_1}(I, a)$.

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent:*
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $j(a) = a$ *and* $a < j(M) < b$.
- **(2)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}(\mathcal{M}, a) = \mathrm{Th}(I, a)$.
- **(3)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}_{\Sigma_1}(\mathcal{M}, a) = \mathrm{Th}_{\Sigma_1}(I, a)$.
- **(4)** $f(a) < b$ *for all partial* $\mathcal{M}$*-recursive functions.*

A Refinement of Hájek-Pudlák's Theorem (1)

A Refinement of Hájek-Pudlák's Theorem (1)

- Let $P \subseteq M$ be a set of parameters. A partial function f from M to M is a *P-partial $\mathcal{M}$-recursive function of $\mathcal{M}$* if the graph of f is definable in $\mathcal{M}$ by a Σ_1-formula with parameters in P.

A Refinement of Hájek-Pudlák's Theorem (1)

- Let $P \subseteq M$ be a set of parameters. A partial function f from M to M is a *P-partial $\mathcal{M}$-recursive function of $\mathcal{M}$* if the graph of f is definable in $\mathcal{M}$ by a Σ_1-formula with parameters in P.
- **Theorem**. *Suppose I is a cut shared by $\mathcal{M}$ and $\mathcal{N}$, and I is closed under exponentiation.*

A Refinement of Hájek-Pudlák's Theorem (1)

- Let $P \subseteq M$ be a set of parameters. A partial function f from M to M is a *P-partial $\mathcal{M}$-recursive function of $\mathcal{M}$* if the graph of f is definable in $\mathcal{M}$ by a Σ_1-formula with parameters in P.
- **Theorem**. *Suppose I is a cut shared by $\mathcal{M}$ and $\mathcal{N}$, and I is closed under exponentiation.*
- *Assume furthermore that $c \in \mathcal{M}$, with $I < c$, and $\{a, b\} \subseteq N$ with $I < a < b$. The following statements are equivalent:*

A Refinement of Hájek-Pudlák's Theorem (1)

- Let $P \subseteq M$ be a set of parameters. A partial function f from M to M is a *P-partial $\mathcal{M}$-recursive function of $\mathcal{M}$* if the graph of f is definable in $\mathcal{M}$ by a Σ_1-formula with parameters in P.
- **Theorem**. *Suppose I is a cut shared by $\mathcal{M}$ and $\mathcal{N}$, and I is closed under exponentiation*.
- *Assume furthermore that $c \in \mathcal{M}$, with $I < c$, and $\{a, b\} \subseteq N$ with $I < a < b$. The following statements are equivalent:*
- **(i)** $\mathrm{SSy}_J(\mathcal{M}) = \mathrm{SSy}_J(\mathcal{N})$, *and for every Δ_0-formula $\delta(x, y, z)$, and all $i \in I$ we have*:

$$\mathcal{M} \models \exists y\ \delta(c, y, i) \Longrightarrow \mathcal{N} \models \exists y < b\ \delta(a, y, i).$$

A Refinement of Hájek-Pudlák's Theorem (1)

- Let $P \subseteq M$ be a set of parameters. A partial function f from M to M is a *P-partial $\mathcal{M}$-recursive function of $\mathcal{M}$* if the graph of f is definable in $\mathcal{M}$ by a Σ_1-formula with parameters in P.
- **Theorem**. *Suppose I is a cut shared by $\mathcal{M}$ and $\mathcal{N}$, and I is closed under exponentiation.*
- *Assume furthermore that $c \in \mathcal{M}$, with $I < c$, and $\{a, b\} \subseteq N$ with $I < a < b$. The following statements are equivalent:*
- **(i)** $\mathrm{SSy}_I(\mathcal{M}) = \mathrm{SSy}_I(\mathcal{N})$, *and for every Δ_0-formula $\delta(x, y, z)$, and all $i \in I$ we have*:

$$\mathcal{M} \models \exists y\ \delta(c, y, i) \Longrightarrow \mathcal{N} \models \exists y < b\ \delta(a, y, i).$$

- **(ii)** *There is an initial embedding $j : \mathcal{M} \to \mathcal{N}$ such that*
  $j(c) = a$,
  $a < j(M) < b$, and
  $j(c) = c$ *for all* $c \in I$.

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $I < a < b$, *where* I *is a cut of* $\mathcal{M}$ *that is closed under exponentiation. The following statements are equivalent:*

Refinement of Hájek-Pudlák's Theorem (2)

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $I < a < b$, *where* I *is a cut of* $\mathcal{M}$ *that is closed under exponentiation. The following statements are equivalent:*

- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$, and $I \cup \{a\} \subseteq \mathrm{Fix}(j)$.

Refinement of Hájek-Pudlák's Theorem (2)

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $I < a < b$, *where* I *is a cut of* $\mathcal{M}$ *that is closed under exponentiation. The following statements are equivalent:*
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$, and $I \cup \{a\} \subseteq \mathrm{Fix}(j)$.
- **(2)** *There is a cut* I^* *of* $\mathcal{M}$ *with* $a < I^* < b$ *and* $\mathrm{Th}\,(\mathcal{M}, a, i)_{i \in I} = \mathrm{Th}(I^*, a, i)_{i \in I}$.

Refinement of Hájek-Pudlák's Theorem (2)

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $I < a < b$, *where* I *is a cut of* $\mathcal{M}$ *that is closed under exponentiation. The following statements are equivalent:*
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$, and $I \cup \{a\} \subseteq \mathrm{Fix}(j)$.
- **(2)** *There is a cut* I^* *of* $\mathcal{M}$ *with* $a < I^* < b$ *and* $\mathrm{Th}\,(\mathcal{M}, a, i)_{i \in I} = \mathrm{Th}(I^*, a, i)_{i \in I}$.
- **(3)** *There is a cut* I^* *of* $\mathcal{M}$ *with* $a < I^* < b$ *such that* $\mathrm{Th}_{\Sigma_1}\,(\mathcal{M}, a, i)_{i \in I} = \mathrm{Th}_{\Sigma_1}(I^*, a, i)_{i \in I}$.

Refinement of Hájek-Pudlák's Theorem (2)

- **Theorem.** *Suppose* $\{a, b\} \subseteq M$ *with* $I < a < b$, *where* I *is a cut of* $\mathcal{M}$ *that is closed under exponentiation. The following statements are equivalent:*
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$, and $I \cup \{a\} \subseteq \mathrm{Fix}(j)$.
- **(2)** *There is a cut* I^* *of* $\mathcal{M}$ *with* $a < I^* < b$ *and* $\mathrm{Th}\,(\mathcal{M}, a, i)_{i \in I} = \mathrm{Th}(I^*, a, i)_{i \in I}$.
- **(3)** *There is a cut* I^* *of* $\mathcal{M}$ *with* $a < I^* < b$ *such that* $\mathrm{Th}_{\Sigma_1}\,(\mathcal{M}, a, i)_{i \in I} = \mathrm{Th}_{\Sigma_1}(I^*, a, i)_{i \in I}$.
- **(4)** $f(a) < b$ *for all* I*-partial* $\mathcal{M}$*-recursive functions* f

The Shavrukov-Wilkie Theorem

The Shavrukov-Wilkie Theorem

- A (total) function f from M to M is a *total $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.

The Shavrukov-Wilkie Theorem

- A (total) function f from M to M is a *total $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem.** *Suppose* $\{a, b\} \subseteq N$ *with* $a < b$. *The following statements are equivalent*:

The Shavrukov-Wilkie Theorem

- A (total) function f from M to M is a *total $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem.** *Suppose* $\{a, b\} \subseteq N$ *with* $a < b$. *The following statements are equivalent*:
- **(1)** $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$, *and for every parameter-free* Δ_0*-formula* $\delta(x, y)$ *we have*:

$$\mathcal{M} \models \forall x \exists y \ \delta(x, y) \Longrightarrow \mathcal{N} \models \exists y < b \ \delta(a, y).$$

The Shavrukov-Wilkie Theorem

- A (total) function f from M to M is a *total $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem.** *Suppose* $\{a, b\} \subseteq N$ *with* $a < b$. *The following statements are equivalent*:
- **(1)** $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$, *and for every parameter-free* Δ_0*-formula* $\delta(x, y)$ *we have*:

$$\mathcal{M} \models \forall x \exists y \; \delta(x, y) \Longrightarrow \mathcal{N} \models \exists y < b \; \delta(a, y).$$

- **(2)** *There is some* $c \in M$ *such that for every parameter-free* Δ_0*-formula* $\delta(x, y)$ *we have:*

$$\mathcal{M} \models \exists y \delta(c, y) \Longrightarrow \mathcal{N} \models \exists y < b \; \delta(a, y).$$

The Shavrukov-Wilkie Theorem

- A (total) function f from M to M is a *total $\mathcal{M}$-recursive function* if the graph of f is definable in $\mathcal{M}$ by a parameter-free Σ_1-formula.
- **Theorem.** *Suppose $\{a, b\} \subseteq N$ with $a < b$. The following statements are equivalent*:
- **(1)** $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$, *and for every parameter-free Δ_0-formula $\delta(x, y)$ we have*:

$$\mathcal{M} \models \forall x \exists y \ \delta(x, y) \Longrightarrow \mathcal{N} \models \exists y < b \ \delta(a, y).$$

- **(2)** *There is some $c \in M$ such that for every parameter-free Δ_0-formula $\delta(x, y)$ we have:*

$$\mathcal{M} \models \exists y \delta(c, y) \Longrightarrow \mathcal{N} \models \exists y < b \ \delta(a, y).$$

- **(3)** *There is an initial embedding $j : \mathcal{M} \to \mathcal{N}$ with $a < j(M) < b$.*

Corollaries of the Shavrukov-Wilkie Theorem

- **Corollary.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent*:

Corollaries of the Shavrukov-Wilkie Theorem

- **Corollary.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent*:
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$.

Corollaries of the Shavrukov-Wilkie Theorem

- **Corollary.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent*:
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$.
- **(2)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(I)$.

Corollaries of the Shavrukov-Wilkie Theorem

- **Corollary.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent*:
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$.
- **(2)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(I)$.
- **(3)** $f(a) < b$ *for all* $\mathcal{M}$*-recursive functions* f.

.

Corollaries of the Shavrukov-Wilkie Theorem

- **Corollary.** *Suppose* $\{a, b\} \subseteq M$ *with* $a < b$. *The following statements are equivalent*:
- **(1)** *There is an initial embedding* $j : \mathcal{M} \to \mathcal{M}$ *with* $a < j(M) < b$.
- **(2)** *There is a cut* I *of* $\mathcal{M}$ *with* $a < I < b$ *and* $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(I)$.
- **(3)** $f(a) < b$ *for all* $\mathcal{M}$*-recursive functions* f.

.

- **Corollary.** $\mathcal{M}$ *is isomorphic to arbitrarily high initial segments of* $\mathcal{N}$ *iff* $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$ and $\mathrm{Th}_{\Pi_2}(\mathcal{M}) \subseteq \mathrm{Th}_{\Pi_2}(\mathcal{N})$.

Tanaka's Theorem (1)

Tanaka's Theorem (1)

- Models of WKL_0 are two-sorted structures of the form $(\mathcal{M}, \mathcal{A})$, where
  $\mathcal{M} = (M, +, \cdot, <, 0, 1) \models \mathrm{I}\Sigma_1$, and

Tanaka's Theorem (1)

- Models of WKL_0 are two-sorted structures of the form $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M} = (M, +, \cdot, <, 0, 1) \models \mathrm{I}\Sigma_1$, and
- $\mathcal{A}$ is a family of subsets of M such that $(\mathcal{M}, \mathcal{A})$ satisfies:
- (1) Induction for Σ_1^0 formulae;

Tanaka's Theorem (1)

- Models of WKL_0 are two-sorted structures of the form $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M} = (M, +, \cdot, <, 0, 1) \models \mathrm{I}\Sigma_1$, and
- $\mathcal{A}$ is a family of subsets of M such that $(\mathcal{M}, \mathcal{A})$ satisfies:
- (1) Induction for Σ_1^0 formulae;
- (2) Comprehension for Δ_1^0-formulae; and

Tanaka's Theorem (1)

- Models of WKL_0 are two-sorted structures of the form $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M} = (M, +, \cdot, <, 0, 1) \models \mathrm{I}\Sigma_1$, and
- $\mathcal{A}$ is a family of subsets of M such that $(\mathcal{M}, \mathcal{A})$ satisfies:
- (1) Induction for Σ^0_1 formulae;
- (2) Comprehension for Δ^0_1-formulae; and
- (3) Weak König's Lemma: every infinite subtree of the full binary tree has an infinite branch.

Tanaka's Theorem (1)

- Models of WKL_0 are two-sorted structures of the form $(\mathcal{M}, \mathcal{A})$, where $\mathcal{M} = (M, +, \cdot, <, 0, 1) \models \mathrm{I}\Sigma_1$, and
- $\mathcal{A}$ is a family of subsets of M such that $(\mathcal{M}, \mathcal{A})$ satisfies:
- (1) Induction for Σ^0_1 formulae;
- (2) Comprehension for Δ^0_1-formulae; and
- (3) Weak König's Lemma: every infinite subtree of the full binary tree has an infinite branch.

- It is well known that every countable model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$ can be expanded to a model $(\mathcal{M}, \mathcal{A}) \models \mathrm{WKL}_0$. This important result is due independently to Harrington and Ratajczyk.

Tanaka's Theorem (2)

Tanaka's Theorem (2)

- **Theorem.** (Tanaka) *Every countable nonstandard model* $(\mathcal{M}, \mathcal{A})$ *of* WKL_0 *is isomorphic to a proper initial segment* I *of itself in the sense that there is an isomorphism* $\phi : \mathcal{M} \to I$ *such that* ϕ *induces an isomorphism* $\widehat{\phi} : (\mathcal{M}, \mathcal{A}) \to (I, \mathcal{A} \upharpoonright I)$.

Tanaka's Theorem (2)

- **Theorem.** (Tanaka) *Every countable nonstandard model* $(\mathcal{M}, \mathcal{A})$ *of* WKL_0 *is isomorphic to a proper initial segment* I *of itself in the sense that there is an isomorphism* $\phi : \mathcal{M} \to I$ *such that* ϕ *induces an isomorphism* $\widehat{\phi} : (\mathcal{M}, \mathcal{A}) \to (I, \mathcal{A} \upharpoonright I)$.
- *Moreover, given any prescribed* $a \in M$, *there is some* I *and* ϕ *as above such that* $\phi(m) = m$ *for all* $m \leq a$.

Tanaka's Theorem (2)

- **Theorem.** (Tanaka) *Every countable nonstandard model* $(\mathcal{M}, \mathcal{A})$ *of* WKL_0 *is isomorphic to a proper initial segment* I *of itself in the sense that there is an isomorphism* $\phi : \mathcal{M} \to I$ *such that* ϕ *induces an isomorphism* $\widehat{\phi} : (\mathcal{M}, \mathcal{A}) \to (I, \mathcal{A} \restriction I)$.
- *Moreover, given any prescribed* $a \in M$, *there is some* I *and* ϕ *as above such that* $\phi(m) = m$ *for all* $m \leq a$.
- $\mathcal{A} \restriction I := \{A \cap I : A \in \mathcal{A}\}$,
  $\widehat{\phi}(m) = \phi(m)$ for $m \in M$,
  and $\widehat{\phi}(A) = \{\phi(a) : a \in A\}$ for $A \in \mathcal{A}$.

Tanaka's Theorem (4)

- **Corollary.** *Every countable nonstandard model* $(\mathcal{M}, \mathcal{A})$ *of* WKL_0 *has an extension* $(\mathcal{M}^*, \mathcal{A}^*)$ *to a model of* WKL_0 *such that* $\mathcal{M}^*$ *properly end extends* $\mathcal{M}$*, and* $\mathcal{A} = \mathcal{A}^* \upharpoonright M$*.*

Tanaka's Theorem (5)

Tanaka's Theorem (5)

- Paris showed that every countable recursively saturated model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ is isomorphic to a proper initial segment of itself, a result that is described by Paris as being 'implicit' in an (unpublished) paper of Solovay.

Tanaka's Theorem (5)

- Paris showed that every countable recursively saturated model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ is isomorphic to a proper initial segment of itself, a result that is described by Paris as being 'implicit' in an (unpublished) paper of Solovay.
- This result of Solovay and Paris can be fine-tuned, as shown by the work of Charalambos Cornaros and Keita Yokoyama (independently).

Tanaka's Theorem (5)

- Paris showed that every countable recursively saturated model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ is isomorphic to a proper initial segment of itself, a result that is described by Paris as being 'implicit' in an (unpublished) paper of Solovay.
- This result of Solovay and Paris can be fine-tuned, as shown by the work of Charalambos Cornaros and Keita Yokoyama (independently).
- **Theorem**. *Suppose $\mathcal{N}$ is a countable model of $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$ that is recursively saturated, and there are $a < b$ in $\mathcal{N}$ such that for every Δ_0-formula $\delta(x, y)$ we have:*

$$\mathcal{N} \models \exists y\ \delta(a, y) \Longrightarrow \mathcal{N} \models \exists y < b\ \delta(a, y).$$

There is an isomorphism $\phi : \mathcal{N} \to I$, where I is an initial segment of $\mathcal{N}$, with
$\phi(a) = a$ and $a < I < b$.

Tanaka's Theorem (6)

Tanaka's Theorem (6)

- **Stage 1:** Given a countable nonstandard model $(\mathcal{M}, \mathcal{A})$ of WKL_0, and a prescribed $a \in M$ in this stage we use the 'muscles' of $\mathrm{I}\Sigma_1$ in the form of the strong Σ_1-collection to locate an element b in $\mathcal{M}$ such that $f(a) < b$ for all partial $\mathcal{M}$-recursive functions f .

Tanaka's Theorem (6)

- **Stage 1:** Given a countable nonstandard model $(\mathcal{M}, \mathcal{A})$ of WKL_0, and a prescribed $a \in M$ in this stage we use the 'muscles' of $\mathrm{I}\Sigma_1$ in the form of the strong Σ_1-collection to locate an element b in $\mathcal{M}$ such that $f(a) < b$ for all partial $\mathcal{M}$-recursive functions f .
- **Stage 2 Outline:** We build an *end extension* $\mathcal{N}$ of $\mathcal{M}$ such that the following conditions hold:
  (I) $\mathcal{N} \models \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$;
  (II) $\mathcal{N}$ is recursively saturated;
  (III) $f(a) < b$ for all partial $\mathcal{N}$-recursive functions; and
  (IV) $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$.

Tanaka's Theorem (6)

- **Stage 1:** Given a countable nonstandard model $(\mathcal{M}, \mathcal{A})$ of WKL_0, and a prescribed $a \in M$ in this stage we use the 'muscles' of $\mathrm{I}\Sigma_1$ in the form of the strong Σ_1-collection to locate an element b in $\mathcal{M}$ such that $f(a) < b$ for all partial $\mathcal{M}$-recursive functions f .
- **Stage 2 Outline:** We build an *end extension* $\mathcal{N}$ of $\mathcal{M}$ such that the following conditions hold:
  (I) $\mathcal{N} \models \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$;
  (II) $\mathcal{N}$ is recursively saturated;
  (III) $f(a) < b$ for all partial $\mathcal{N}$-recursive functions; and
  (IV) $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$.
- **Stage 3 Outline**: We use the refined Paris-Solovay theorem to embed $\mathcal{N}$ onto a proper initial segment J of $\mathcal{M}$. By elementary considerations, this will yield a proper cut I of J with $(\mathcal{M}, \mathcal{A}) \cong (I, \mathcal{A} \restriction I)$.

Tanaka's Theorem (7)

Tanaka's Theorem (7)

- Stage 2 of the proof can be summarized into a theorem in its own right.

Tanaka's Theorem (7)

- Stage 2 of the proof can be summarized into a theorem in its own right.
- **Theorem**. *Let* $(\mathcal{M}, \mathcal{A})$ *be a countable model of* WKL_0 *and let* $b \in M$. *Then* $\mathcal{M}$ *has a recursively saturated proper end extension* $\mathcal{N}$ *satisfying* $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathrm{PRA}$ *such that* $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$, *and* $\mathcal{N}$ *is a conservative extension of* $\mathcal{M}$ *with respect to* $\Pi_{1,\leq b}$*-sentences.*

Tanaka's Theorem (7)

- Stage 2 of the proof can be summarized into a theorem in its own right.
- **Theorem**. *Let* $(\mathcal{M}, \mathcal{A})$ *be a countable model of* WKL_0 *and let* $b \in M$*. Then* $\mathcal{M}$ *has a recursively saturated proper end extension* $\mathcal{N}$ *satisfying* $\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathsf{PRA}$ *such that* $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$*, and* $\mathcal{N}$ *is a conservative extension of* $\mathcal{M}$ *with respect to* $\Pi_{1,\leq b}$*-sentences.*
- Follow Scott's strategy of showing "countable Scott sets can be realized as the standard system of a (recursively saturated) model of PA";
- (Beklemishev, 1998; refining Clote-Hájek-Paris, 1990)

  $\mathrm{I}\Sigma_1 \vdash \mathsf{Con}(\mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1 + \mathbf{True}_{\Pi_2})$.

A Characterization of WKL_0

A Characterization of WKL_0

- **Theorem**. *Let* $(\mathcal{M}, \mathcal{A})$ *be a countable model of* RCA_0. *The following are equivalent*:
  (1) $(\mathcal{M}, \mathcal{A})$ *is a model of* WKL_0.

- **Theorem**. *Let* $(\mathcal{M}, \mathcal{A})$ *be a countable model of* RCA_0. *The following are equivalent*:
  (1) $(\mathcal{M}, \mathcal{A})$ *is a model of* WKL_0.
- **(2)** *For every* $b \in M$ *there exists a recursively saturated proper end extension* $\mathcal{N}$ *of* $\mathcal{M}$ *such that* $\mathcal{N} \models \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$, $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$, *and* $\mathcal{N}$ *is a conservative extension of* $\mathcal{M}$ *with respect to* $\Pi_{1,\leq b}-$*sentences.*

A Characterization of WKL_0

- **Theorem**. *Let* $(\mathcal{M}, \mathcal{A})$ *be a countable model of* RCA_0. *The following are equivalent*:
  (1) $(\mathcal{M}, \mathcal{A})$ *is a model of* WKL_0.
- **(2)** *For every* $b \in M$ *there exists a recursively saturated proper end extension* $\mathcal{N}$ *of* $\mathcal{M}$ *such that* $\mathcal{N} \models \mathrm{I}\Delta_0 + \mathrm{B}\Sigma_1$, $\mathrm{SSy}_M(\mathcal{N}) = \mathcal{A}$, *and* $\mathcal{N}$ *is a conservative extension of* $\mathcal{M}$ *with respect to* $\Pi_{1,\leq b}$*−sentences.*
- **(3)** *For every* $b \in M$ *there is a proper initial segment* I *of* $\mathcal{M}$ *such that* $(\mathcal{M}, \mathcal{A})$ *is isomorphic to* $(I, \mathcal{A} \restriction I)$ *via an isomorphism that pointwise fixes* $M_{\leq b}$.

Controlling Fixed Points (1)

Controlling Fixed Points (1)

- **Theorem.** (E, 2012). *Suppose* I *is a proper cut of* $\mathcal{M}$ *that is closed under exponentiation. There is a* Σ_1*-elementary extension* $\mathcal{N}$ *of* $\mathcal{M}$ *such that* $\mathrm{SSy}_I(\mathcal{N}) = \mathrm{SSy}_I(\mathcal{M})$ *and* $\mathrm{I_{fix}}(j) = I$ *for some* $j \in \mathrm{Aut}(\mathcal{N})$.

Controlling Fixed Points (1)

- **Theorem.** (E, 2012). *Suppose* I *is a proper cut of* $\mathcal{M}$ *that is closed under exponentiation. There is a* Σ_1*-elementary extension* $\mathcal{N}$ *of* $\mathcal{M}$ *such that* $\mathrm{SSy}_I(\mathcal{N}) = \mathrm{SSy}_I(\mathcal{M})$ *and* $\mathrm{I_{fix}}(j) = I$ *for some* $j \in \mathrm{Aut}(\mathcal{N})$.
- **Theorem.** (E, 2012) *Suppose* I *is proper cut of* $\mathcal{M}$. *The following conditions are equivalent.*

Controlling Fixed Points (1)

- **Theorem.** (E, 2012). *Suppose I is a proper cut of $\mathcal{M}$ that is closed under exponentiation. There is a Σ_1-elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that* $\mathrm{SSy}_I(\mathcal{N}) = \mathrm{SSy}_I(\mathcal{M})$ *and* $\mathrm{I}_{\mathrm{fix}}(j) = I$ *for some* $j \in \mathrm{Aut}(\mathcal{N})$.
- **Theorem.** (E, 2012) *Suppose I is proper cut of $\mathcal{M}$. The following conditions are equivalent.*
- **(1)** *There is an initial self-embedding $j : \mathcal{M} \to \mathcal{M}$ such that* $\mathrm{I}_{\mathrm{fix}}(j) = I$.

Controlling Fixed Points (1)

- **Theorem.** (E, 2012). *Suppose I is a proper cut of $\mathcal{M}$ that is closed under exponentiation. There is a Σ_1-elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathrm{SSy}_I(\mathcal{N}) = \mathrm{SSy}_I(\mathcal{M})$ and $\mathrm{I}_{\mathrm{fix}}(j) = I$ for some $j \in \mathrm{Aut}(\mathcal{N})$.*
- **Theorem.** (E, 2012) *Suppose I is proper cut of $\mathcal{M}$. The following conditions are equivalent.*
- **(1)** *There is an initial self-embedding $j : \mathcal{M} \to \mathcal{M}$ such that $\mathrm{I}_{\mathrm{fix}}(j) = I$.*
- **(2)** *I is closed under exponentiation.*

Controlling Fixed Points (2)

Controlling Fixed Points (2)

- **Theorem**. (E, 2012) *Suppose I is proper initial segment of $\mathcal{M}$. The following conditions are equivalent.*

Controlling Fixed Points (2)

- **Theorem**. (E, 2012) *Suppose I is proper initial segment of $\mathcal{M}$. The following conditions are equivalent.*
- **(1)** *There is an initial self-embedding $j : \mathcal{M} \to \mathcal{M}$ such that* $\mathrm{Fix}(j) = I$.

Controlling Fixed Points (2)

- **Theorem**. (E, 2012) *Suppose I is proper initial segment of $\mathcal{M}$. The following conditions are equivalent.*
- **(1)** *There is an initial self-embedding $j : \mathcal{M} \to \mathcal{M}$ such that* $\mathrm{Fix}(j) = I$.
- **(2)** *I is a strong cut of $\mathcal{M}$, and $I \prec_{\Sigma_1} \mathcal{M}$.*

Controlling Fixed Points (3)

Controlling Fixed Points (3)

- **Theorem**. (E, 2012) *The following conditions are equivalent.*
  (1) *There is an initial self-embedding* $j : \mathcal{M} \to \mathcal{M}$ *such that* $\mathrm{Fix}(j) = K^1(\mathcal{M})$.
  (2) $\mathbb{N}$ *is a strong cut of* $\mathcal{M}$.

The Germomorphism Group (1)

The Germomorphism Group (1)

- Suppose I is a proper cut of $\mathcal{M}$. Let $G_I(\mathcal{M})$ be the group of "I-endomorphism of $\mathcal{M}$" whose elements are E-equivalence classes of isomorphisms between initial segments of $\mathcal{M}$ that properly contain I and which pointwise fix I, where E stands for "agreement on a cut that properly extends I".

The Germomorphism Group (1)

- Suppose I is a proper cut of $\mathcal{M}$. Let $G_I(\mathcal{M})$ be the group of "I-endomorphism of $\mathcal{M}$" whose elements are E-equivalence classes of isomorphisms between initial segments of $\mathcal{M}$ that properly contain I and which pointwise fix I, where E stands for "agreement on a cut that properly extends I".
- **Theorem.** (E, 2012) *There is an embedding of* $\mathrm{Aut}(\mathbb{Q})$ *into* $G(\mathcal{M})$.

The Germomorphism Group (2)

The Germomorphism Group (2)

- Key technical device:
  Theorem. *Let J be a proper cut of $\mathcal{M}$ that is closed under exponentiation. There is an* **elementary** *embedding λ from $\mathcal{M}$ into $\mathcal{M}^*$, and an embedding $\psi \mapsto \widehat{\psi}$ from $\mathrm{Aut}(\mathbb{Q})$ to $\mathrm{Aut}(\mathcal{M}^*)$ such that:*

The Germomorphism Group (2)

- Key technical device:
  Theorem. *Let J be a proper cut of $\mathcal{M}$ that is closed under exponentiation. There is an* **elementary** *embedding λ from $\mathcal{M}$ into $\mathcal{M}^*$, and an embedding $\psi \mapsto \widehat{\psi}$ from $\mathrm{Aut}(\mathbb{Q})$ to $\mathrm{Aut}(\mathcal{M}^*)$ such that:*

- **(1)** $\lambda(\mathcal{M}) \prec_{\text{cofinal}} \mathcal{M}^*$;

The Germomorphism Group (2)

- Key technical device:
  Theorem. *Let J be a proper cut of $\mathcal{M}$ that is closed under exponentiation. There is an* **elementary** *embedding λ from $\mathcal{M}$ into $\mathcal{M}^*$, and an embedding $\psi \mapsto \widehat{\psi}$ from $\mathrm{Aut}(\mathbb{Q})$ to $\mathrm{Aut}(\mathcal{M}^*)$ such that:*
- **(1)** $\lambda(\mathcal{M}) \prec_{\mathrm{cofinal}} \mathcal{M}^*$;
- **(2)** *J is the longest initial segment of $\mathcal{M}$ that is pointwise fixed by λ;*

The Germomorphism Group (2)

- Key technical device:
  Theorem. *Let J be a proper cut of $\mathcal{M}$ that is closed under exponentiation. There is an* **elementary** *embedding λ from $\mathcal{M}$ into $\mathcal{M}^*$, and an embedding $\psi \mapsto \widehat{\psi}$ from $\mathrm{Aut}(\mathbb{Q})$ to $\mathrm{Aut}(\mathcal{M}^*)$ such that:*
- **(1)** $\lambda(\mathcal{M}) \prec_{\mathrm{cofinal}} \mathcal{M}^*$;
- **(2)** *J is the longest initial segment of $\mathcal{M}$ that is pointwise fixed by λ;*
- **(3)** $\mathrm{SSy}_J(\mathcal{M}) = \mathrm{SSy}_J(\mathcal{M}^*)$;

The Germomorphism Group (2)

- Key technical device:
  Theorem. *Let J be a proper cut of $\mathcal{M}$ that is closed under exponentiation. There is an* **elementary** *embedding λ from $\mathcal{M}$ into $\mathcal{M}^*$, and an embedding $\psi \mapsto \widehat{\psi}$ from $\mathrm{Aut}(\mathbb{Q})$ to $\mathrm{Aut}(\mathcal{M}^*)$ such that:*

- **(1)** $\lambda(\mathcal{M}) \prec_{\mathrm{cofinal}} \mathcal{M}^*$;
- **(2)** *J is the longest initial segment of $\mathcal{M}$ that is pointwise fixed by λ;*
- **(3)** $\mathrm{SSy}_J(\mathcal{M}) = \mathrm{SSy}_J(\mathcal{M}^*)$;
- **(4)** *$\widehat{\psi}(m) = m$ for all $m \in \mathcal{M}$ and all $\psi \in \mathrm{Aut}(\mathbb{Q})$;*

The Germomorphism Group (2)

- Key technical device:
  Theorem. *Let* J *be a proper cut of* $\mathcal{M}$ *that is closed under exponentiation. There is an* **elementary** *embedding* λ *from* $\mathcal{M}$ *into* $\mathcal{M}^*$, *and an embedding* $\psi \mapsto \widehat{\psi}$ *from* $\mathrm{Aut}(\mathbb{Q})$ *to* $\mathrm{Aut}(\mathcal{M}^*)$ *such that:*

- **(1)** $\lambda(\mathcal{M}) \prec_{\mathrm{cofinal}} \mathcal{M}^*$;
- **(2)** *J is the longest initial segment of $\mathcal{M}$ that is pointwise fixed by λ;*
- **(3)** $\mathrm{SSy}_J(\mathcal{M}) = \mathrm{SSy}_J(\mathcal{M}^*)$;
- **(4)** $\widehat{\psi}(m) = m$ *for all* $m \in \mathcal{M}$ *and all* $\psi \in \mathrm{Aut}(\mathbb{Q})$;
- **(5)** $J = I_{\mathrm{fix}}(\widehat{\psi}) := \{x \in M^* : \forall y \leq x \ \widehat{\psi}(y) = y\}$ *for all* $\psi \in \mathrm{Aut}(\mathbb{Q})$; *and*

The Germomorphism Group (2)

- Key technical device:
  Theorem. *Let J be a proper cut of $\mathcal{M}$ that is closed under exponentiation. There is an* **elementary** *embedding λ from $\mathcal{M}$ into $\mathcal{M}^*$, and an embedding $\psi \mapsto \widehat{\psi}$ from $\mathrm{Aut}(\mathbb{Q})$ to $\mathrm{Aut}(\mathcal{M}^*)$ such that:*

- **(1)** $\lambda(\mathcal{M}) \prec_{\text{cofinal}} \mathcal{M}^*$;
- **(2)** *J is the longest initial segment of $\mathcal{M}$ that is pointwise fixed by λ;*
- **(3)** $\mathrm{SSy}_J(\mathcal{M}) = \mathrm{SSy}_J(\mathcal{M}^*)$;
- **(4)** $\widehat{\psi}(m) = m$ *for all* $m \in \mathcal{M}$ *and all* $\psi \in \mathrm{Aut}(\mathbb{Q})$;
- **(5)** $J = I_{\text{fix}}(\widehat{\psi}) := \{x \in M^* : \forall y \leq x\ \widehat{\psi}(y) = y\}$ *for all* $\psi \in \mathrm{Aut}(\mathbb{Q})$; *and*
- **(6)** $\left\{m \in \mathcal{M}^* : \widehat{\psi}_1(m) \neq \widehat{\psi}_2(m)\right\}$ *is downward cofinal in* $\mathcal{M}^* \backslash J$ *for all distinct* ψ_1 *and* ψ_2 *in* $\mathrm{Aut}(\mathbb{Q})$.

Questions About the Germomorphism Group

Questions About the Germomorphism Group

- **(1)** Is there an uncountable M for which $G(\mathcal{M})$ is the trivial group?

Questions About the Germomorphism Group

- **(1)** Is there an uncountable M for which $G(\mathcal{M})$ is the trivial group?
- **(2**) Suppose $\mathcal{M}$ **is recursively saturated.** What is the relationship between $\mathrm{Aut}(\mathcal{M})$and $G(\mathcal{M})$?

Questions About the Germomorphism Group

- **(1)** Is there an uncountable M for which $G(\mathcal{M})$ is the trivial group?
- **(2)** Suppose $\mathcal{M}$ **is recursively saturated.** What is the relationship between $\mathrm{Aut}(\mathcal{M})$and $G(\mathcal{M})$?
- **(3)** What kind of groups can arise as $G(\mathcal{M})$?