Reciprocity laws and Δ_0-definability

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Plan

- What is Δ_0-definability ?

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- Dedekind sums are Δ_0-definable

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- Reciprocity schema

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- Reciprocity schema
- Conclusion

What is Δ_0−definability ?

x is not prime nor 0 nor 1

$$(\exists u) \qquad (\exists v) \qquad (x = uv) \wedge (u \neq x) \wedge (u \neq x)$$

What is Δ_0–definability ?

x is not prime nor 0 nor 1

$$(\exists u)_{u<x}\,(\exists v)_{v<x}\,(x = uv) \wedge (u \neq x) \wedge (v \neq x)$$

What is Δ_0−definability ?

Major open problem

Find a "simple" artithmetical relation

NOT Δ_0−definable

What is Δ_0-definability ?

Exemple

$$z = x^y$$

What is Δ_0−definability ?

Exemple

$$z = x^y$$

$y = 2 \wedge z = x.x$ and $y = 3 \wedge z = x.x.x$ and ...

What is Δ_0-definability ?

Exemple

$$z = x^x$$

IS Δ_0-definable

What is Δ_0-definability ?

Open exemple

z is the n-th prime number

IS NOT KNOWN TO BE Δ_0-definable

A new Δ_0-definable relation and a standard method

Let d and c be integers, $c \neq 0$. The Dedekind sum is defined by

$$s(d,c) = \sum_{k=1}^{k=|c|} \left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{kd}{c}\right)\right)$$

where $((x)) = 0 \text{ if } x \in Z$, else $x - [x] - \frac{1}{2}$.

A new Δ_0−definable relation and a standard method

$s(d,c)$ is a rational number, but

$$12c \times s(d,c) \text{ is a rational integer.}$$

A new Δ_0-definable relation and a standard method

$s(d, c)$ is a rational number, but

$$12c \times s(d, c) \text{ is a rational integer.}$$

$$s(d, c) = \frac{c-1}{12c}(4cd - 2d - 3c) - \frac{1}{c}\sum_{n=1}^{c-1} n \left\lfloor \frac{dn}{c} \right\rfloor$$

A new Δ_0−definable relation and a standard method

Theorem : $12c \times s(d, c)$ is Δ_0−definable.

A new Δ_0−definable relation and a standard method

Lemma (Ph. Barkan, 1977)

$$s(d,c) = \frac{1}{12}\left(-3 + \frac{d + d^{-1} \bmod c}{c} - \sum_{i=1}^{i=r}(-1)^i a_i\right)$$

where $(a_i)_{0\le i\le r}$ is the sequence of the continued fraction development

$$\frac{d}{c} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{\cdots}{\cdots + \frac{1}{a_r}}}}$$

A new Δ_0-definable relation and a standard method

Lemma (Woods, 1981)

$$z = \sum_{i=0}^{i=g(\mathbf{x})} f(i, \mathbf{x}) \text{ is } \Delta_0-\text{definable provided}$$

• the graphs of $f(i, \mathbf{x})$ and $g(\mathbf{x})$ are Δ_0- definable
• f is polynomially bounded
• g is polylogarithmicaly bounded (i.e. there exists a polynomial function ψ with positive integer coefficients, such that $g(\mathbf{x}) \leq \psi(\lfloor log_2(x_1)\rfloor, ..., \lfloor log_2(x_k)\rfloor))$

A new Δ_0-definable relation and a standard method

Lemma (H.-A. E. 2010)

Let c and d be two positive integers. The sequence of the coefficients $(a_i(d,c))_{0\leq i\leq r(d,c)}$ of the continued fraction development of $\frac{d}{c}$ is Δ_0-definable.

Generalization of Dedekind sum

Main result

Generalization of Dedekind sum

Let us consider now the following generalization known (?) as the Rademacher-Dieter-Knuth-Dedekind sum :

$$r_u(d,c) = \sum_{k=1}^{k=|c|} \left(\left(\frac{k}{c}\right)\right) \left(\left(\frac{kd+u}{c}\right)\right)$$

where $u \in Z$.

Generalization of Dedekind sum

$r_u(d, c)$ is a rational number, but

$$12c \times r_u(d, c) \text{ is a rational integer.}$$

Theorem : $12c \times r_u(d, c)$ is Δ_0-definable.

Generalization of Dedekind sum

Reciprocity law (U. Dieter, 1959)

$$r_u(d,c)+r_u(c,d) = \frac{1}{12}\left(\frac{d}{c}+\frac{c}{d}+\frac{1+6\lfloor u\rfloor\lceil u\rceil}{cd} - 6\lfloor\frac{u}{c}\rfloor - 3e(d,u)\right)$$

for $d > u > 0$ and $d \geq c > 0$ and $e(d,u) = 0$ if $u > 0$ and $u \equiv 0 \mod c$, else $e(c,u) = 1$.

Generalization of Dedekind sum

$r_0 = c \qquad r_1 = d \qquad u_0 = u$

Euclidean algorithm

$r_{i+2} = r_i \bmod r_{i+1} \qquad u_{i+1} = u_i \bmod r_{i+1}$

$\alpha_i = \lfloor \frac{r_i}{r_{i+1}} \rfloor \qquad \beta_i = \lfloor \frac{u_i}{r_{i+1}} \rfloor$

$r_{l-1} = \alpha_{l-1} r_l \qquad u_{l-1} = \beta_{l-1} r_l + u_l$

Generalization of Dedekind sum

$p_0 = a_0 \qquad p_1 = a_0 a_1 + 1 \qquad q_0 = 1 \qquad q_1 = a_1$

$\alpha_i p_{i-1} + p_{i-2} = p_i \qquad \alpha_i q_{i-1} + q_{i-2} = q_i$

Generalization of Dedekind sum

Algorithm (D. E. Knuth, 1977)

$$12c \times r_u(d,c) = c\left(\sum_{j=0}^{j=l-1}(-1)^j\left(\alpha_j - 6\beta_j - 3e(r_{j+1},u_j)\right)\right)$$

$$+\left(d+(-1)^{l-1}p_l + 6\sum_{j=0}^{j=l-1}(-1)^j\beta_j(u_j+u_{j+1})p_{j+1}\right)$$

Generalization of Dedekind sum

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Generalization of Dedekind sum

The idea is to code $(\beta_j)_{0\leq j\leq l}$ with (Γ_1, Γ_2) such that

$$\frac{\Gamma_1}{\Gamma_2} = \beta_0 + \frac{1}{\beta_1 + \frac{\dots}{\dots + \frac{1}{\beta_{l-1}}}}$$

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The code is polynomially bounded :
$\Gamma_1 < ud$ and $\Gamma_2 < d$.

Generalization of Dedekind sum

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The decoding function $\alpha_j(\Gamma_1, \Gamma_2)$ is rudimentary.

Generalization of Dedekind sum

$z = \beta_j$ is definned by

$$\exists\,(\Gamma_1)_{<ud}\,\exists\,(\Gamma_2)_{<d}\,\exists\,(\Gamma_3)_{<u}\left(z = \alpha_j(\Gamma_1, \Gamma_2)\right) \wedge$$

$$\left(\forall i \left(\Gamma_3 + \sum_{k=l-1}^{k=i} \alpha_j(u, v) r_{i+1} < r_i\right)\right)$$

and simillary for $z = u_j$.

Reciprocity laws

RL schema :

$$\begin{cases} f(a,b,\vec{x}) = f(a-b,b,\vec{x}) \text{ if a > b} \\ f(a,0,\vec{x}) = g(a,\vec{x}) \\ f(a,b,\vec{x}) = h(f(b,a,\vec{x}),a,b,\vec{x}) \\ f(a,b,\vec{x}) \leq poly(a,b,\vec{x})?? \end{cases}$$

defines f from g and h with N as domain and codomain.

Reciprocity laws

summing RL schema :

$$\begin{cases} f(a,b,\vec{x}) = f(a-b,b,\vec{x}) \text{ if a > b} \\ f(a,0,\vec{x}) = g(a,\vec{x}) \\ f(a,b,\vec{x}) + f(b,a,\vec{x}) = h(a,b,\vec{x}) \\ \mathbf{h}(a,b,\vec{x}) \leq poly(a,b,\vec{x}) \end{cases}$$

defines *f* from *g* and *h* with N as domain and codomain.

Reciprocity laws

Proposition : The set of rudimentary functions is closed under the RL summing schema.

Reciprocity laws

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Proof :

$$f(a,b,\vec{x}) = \sum_{j=1}^{j=l} (-1)^{j+1} h(r_{j+1}, r_j, \vec{x}) + (-1)^l g(r_l, \vec{x})$$

$r_0 = a$ $\qquad r_1 = b$ $\qquad$ Euclidean algorithm

$r_{i+2} = r_j \bmod r_{i+1}$ $\qquad r_{l-1} = \alpha_{l-1} r_l$

$r_l = gcd(a,b)$

Reciprocity laws

	Recip. Law schema	log. long pol. bounded rec. schema	classical pol bounded rec. schema
complete	?	?	?
sum weakened	closed	closed	equiv. counting

Reciprocity laws

	Recip. Law schema	log. long pol. bounded rec. schema	classical pol bounded rec. schema
complete	←	← ←	←
sum weakened	closed	closed	equiv. counting

Conclusion and further work

More generalization of Dedekind sum

$$s(a,b,c) = \sum_{k=1}^{k=c} \left(\left(\frac{ak}{c}\right)\right)\left(\left(\frac{bk}{c}\right)\right)$$

Conclusion and further work

Reciprocity (Rademacher, 1954)

$$s(a,b,c) + s(b,c,a) + s(c,a,b) = \frac{1}{4} - \frac{1}{12}\left(\frac{c}{ab} + \frac{a}{bc} + \frac{b}{ac}\right)$$

Conclusion and further work

Conjecture : it is $\Delta_0^\sharp$ definable