The generic choice of a cut

Tin Lok Wong

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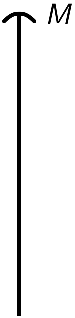
Cuts



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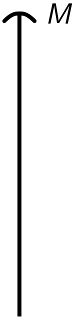

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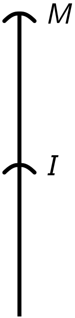

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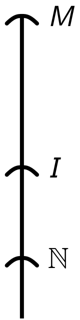

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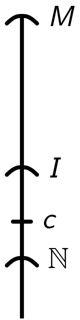

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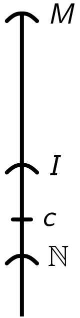

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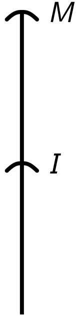

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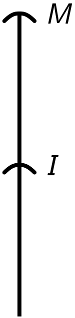

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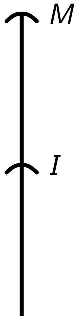

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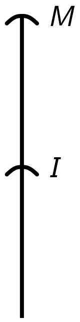

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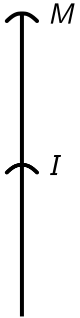

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- Reverse mathematics

Generic cuts (Kaye 2008)

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What do generic cuts look like?

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Plan

- Topological definition
- The opposite of being special
- Functions under which the cut is closed
- Model theoretic properties

Indicators

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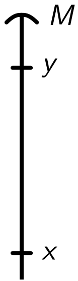

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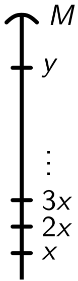

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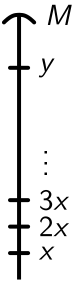

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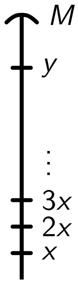

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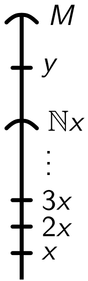

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Proposition (Kotlarski 1984)

$\mathscr{C}$ is homeomorphic to the Cantor space 2^ω.

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The set of *generic cuts* is this smallest comeagre set.

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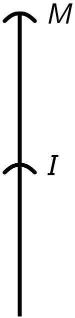

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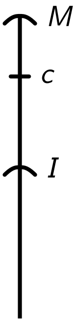

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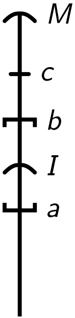

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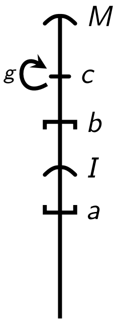

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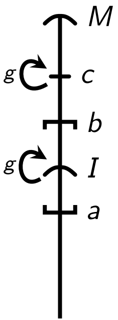

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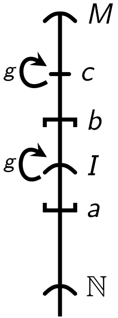

Example

$\mathbb{N}$ is a special cut over every $c \in M$.

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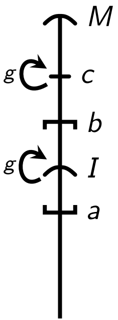

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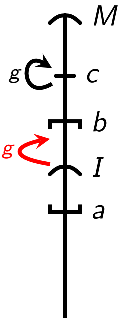

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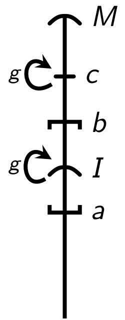

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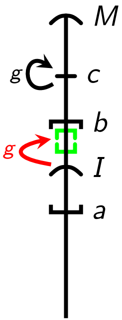

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Theorem (Kaye–W 2010)

A Y-cut I is generic if and only if

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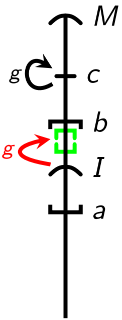

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A generic cut can move about freely in its neighbourhood.

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Every $\exists$ formula in $\mathscr{L}_{\text{cut}}$ is equivalent over PA_Y to

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for some Skolem function F.

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Theorem (Kaye–W 2010 & 2013$^+$, W)

A Y-cut I is generic if and only if the following hold.

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is bounded below in $M \setminus I$.

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Genericity is a robust notion!