End-extensions of models of second-order arithmetic

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Plan

- First-order arithmetic
- Second-order arithmetic
- The Big Five and beyond

First-order arithmetic

- $\mathscr{L}_{\mathrm{I}} = \{0, 1, +, \times, <, =\}$.
- Quantifiers of the forms $\forall x{<}t$ and $\exists x{<}t$ are *bounded*.
- $\mathscr{L}_{\mathrm{I}}$-formulas in which all quantifiers are bounded are Δ_0.
- Σ_n*-formulas* are $\mathscr{L}_{\mathrm{I}}$-formulas of the form ($n \in \mathbb{N}$)

  $$\exists \bar{x}_1 \ \forall \bar{x}_2 \ \cdots \ \mathrm{Q}\bar{x}_n \ \varphi(\bar{x}, \bar{z}),$$

  where $\varphi \in \Delta_0$ and $\mathrm{Q} \in \{\forall, \exists\}$.
- Π_n*-formulas* are negations of Σ_n-formulas. ($n \in \mathbb{N}$)
- $\mathrm{I}\Gamma$ consists of PA^- and

  $$\theta(0) \wedge \forall x \ (\theta(x) \to \theta(x+1)) \to \forall x \ \theta(x)$$

  for all $\theta \in \Gamma$. (Γ is a set of $\mathscr{L}_{\mathrm{I}}$-formulas.)
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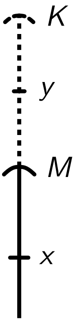
End-extensions and collection



Definition

An *end-extension* of $M \models \mathrm{PA}^-$ is $K \supseteq M$ such that

$$\forall x \in M \;\; \forall y \in K \setminus M \;\; x < y.$$

Definition

For each $n \in \mathbb{N}$, we have $\mathrm{B}\Sigma_n$ axiomatized by $\mathrm{I}\Delta_0$ and

$$\forall a \; \big(\forall x{<}a \;\; \exists y \;\; \theta(x,y) \rightarrow \exists b \;\; \forall x{<}a \;\; \exists y{<}b \;\; \theta(x,y)\big),$$

where θ ranges over Σ_n.

Proposition (Parsons 1970, Paris–Kirby 1978)

$\mathrm{I}\Sigma_{n+1} \vdash \mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$ for all $n \in \mathbb{N}$.

Model theory of collection

Theorem (Paris–Kirby 1978)

For a countable $M \models \mathrm{I}\Delta_0$ and $n \geqslant 2$, the following are equivalent.

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Theorem (folkore?)

If M is a countable model of $\mathrm{B}\Sigma_1 + \mathrm{exp}$, then M has a proper end-extension $K \models \mathrm{I}\Delta_0$.

Model theory of induction

Theorem (Mac Dowell–Specker 1961, Paris–Kirby 1978)

For $M \models \mathrm{I}\Delta_0$, the following are equivalent.

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Proof

Self-embed M. □

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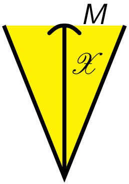
- So $\mathscr{L}_{\mathrm{II}}$-structures are of the form $(M, \mathscr{X})$, where $\mathscr{X} \subseteq \mathcal{P}(M)$.



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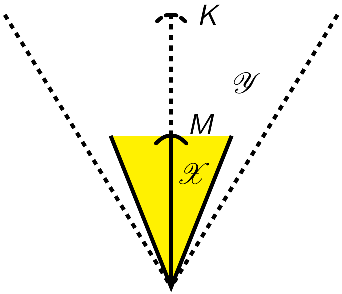

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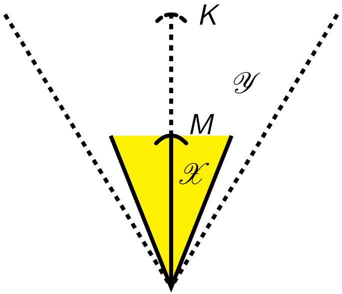

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In second-order arithmetic: $\mathscr{X} \subseteq \mathcal{P}(M)$ and $\mathscr{Y} \subseteq \mathcal{P}(K)$.
In model theory: $M \subseteq K$ and $\mathscr{X} \subseteq \mathscr{Y}$.

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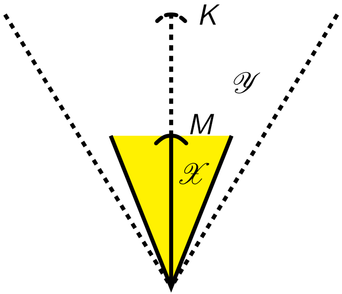

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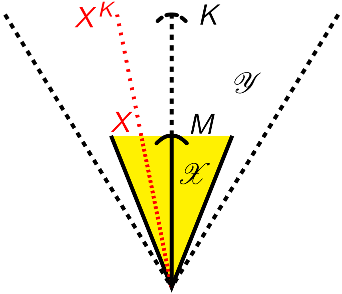
Solution

Make explicit an embedding $\mathscr{X} \hookrightarrow \mathscr{Y}$

$$X \mapsto X^K.$$



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Subsystems of second-order arithmetic

- For a class Γ of $\mathscr{L}_{\mathrm{II}}$-formulas, define

$$\Gamma\text{-CA} = \left\{ \, \exists X \; \forall v \; \big(v \in X \leftrightarrow \theta(v) \big) \; : \; \theta \in \Gamma \, \right\}.$$

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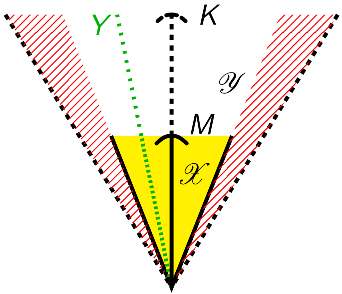

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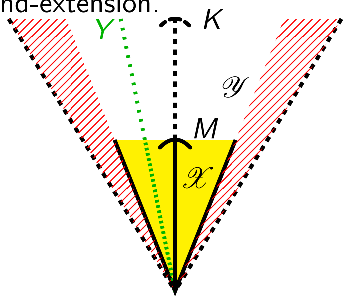

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(a) $(M, \mathscr{X}) \models \mathsf{ACA}_0$.

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Yokoyama 2007: Σ^1_1-elementary

More elementarity

Theorem (Yokoyama)

Let $n \geqslant 1$. Then every countable model of $\mathsf{RCA}_0 + \Sigma^1_n\text{-}\mathsf{CA} + \Sigma^1_n\text{-}\mathsf{AC}$ has a proper Σ^1_{n+1}-elementary conservative end-extension.

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Every countable model of $\Pi^1_1\text{-}\mathsf{CA}_0$ has a proper Σ^1_2-elementary conservative end-extension.

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$\Pi^1_1\text{-}\mathsf{CA}_0 \vdash \mathsf{RCA}_0 + \Sigma^1_1\text{-}\mathsf{CA} + \Sigma^1_1\text{-}\mathsf{AC}$. □

The Axiom of Choice in second-order arithmetic

Definition

For a class of $\mathscr{L}_{\mathrm{II}}$-formulas Γ, define Γ-AC to be the set of all sentences of the form

$$\forall x\ \exists Y\ \theta(x, Y) \rightarrow \exists Y\ \forall x\ \theta(x, (Y)_x)$$

where $\theta \in \Gamma$. Here $(Y)_x = \{y : \langle x, y \rangle \in Y\}$.

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Notation

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Theorem (Kaye–W)

For a countable $(M, \mathscr{X}) \models \mathsf{RCA}_0$, the following are equivalent.

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Combinatorial basis

Theorem (Friedman–McAloon–Simpson 1982)

ATR_0 is equivalent over RCA_0 to

$$\forall^{\mathsf{cf}} S \;\; \exists H \subseteq_{\mathsf{cf}} S \;\; \big(\forall X \subseteq_{\mathsf{cf}} H \;\; \xi(X) \vee \forall X \subseteq_{\mathsf{cf}} H \;\; \neg\xi(X)\big),$$

where ξ ranges over Σ^0_1.

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Theorem (Simpson)

Π^1_1-CA_0 is equivalent over RCA_0 to

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where ζ ranges over arithmetical formulas.

Concluding questions

(1) To what extent are the following pairs similar?

$$\begin{array}{lcl} \mathrm{B}\Sigma_{n+1} & \sim & \Sigma^1_n\text{-CA} + \Sigma^1_n\text{-AC} \\ \mathrm{B}\Sigma_1 & \sim & \mathrm{ATR}_0 \\ \mathrm{B}\Sigma_1 + \exp & \sim & \mathrm{ATR}_0 + \Sigma^0_1\text{-RT} \end{array}$$

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(2) Are the similarities merely superficial?

(3) What is the role played by definable types in second-order arithmetic?

An excursion for me

Theorem

For a countable $M \models \mathrm{I}\Delta_0$, the following are equivalent.

(a) $M \models \mathrm{B}\Sigma_1$.

(b) M has a Σ_2-elementary *cofinal* extension K containing some g such that

  (i) for all Σ_1- and Π_1-formulas $\xi(x)$,

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Notation

- $\mathrm{Def}^*_{\Pi_1}(M) = \{A \in \mathrm{Def}_{\Pi_1}(M) : A \text{ is bounded and infinite}\}$.
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(ii) for all $A \in \mathrm{Filt}_M(g)$ and $\theta \in \Delta_0$, there is $b \in M$ such that

$$A \Vdash \exists v \; \theta(x, v) \quad \Rightarrow \quad A \Vdash \exists v{<}b \;\; \theta(x, v).$$

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